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REFLECTION OF A PLANE LONGITUDINAL SHOCK WAVE OF CONSTANT INTENSITY FROM A PLANE RIGID BOUNDARY WITH A NONLINEAR ELASTIC MEDIUM

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UDC 539.3

A consequence of the second law of thermodynamics in gasdynamics is the well-known theorem of Cemplen on the existence of only compressional shock waves. Ths system of differential equations of gasdynamics has the property that they lead to solutions consistent with this theorem. With certain additional conditions, a similar situations occurs for quasi-longitudinal (bulk) shock waves in an elastic medium. In particular, in the formulation of self-modeling problems in the nonlinear dynamical theory of elasticity [1], one can often prove a priori that the leading front of bulk deformations propagating in the elastic medium is either a shock wave or a centralized wave depending on whether the introduced perturbations lead to compression or expansion of the medium. Another case is that of quasitransverse (shear) shock waves. We note that [2] a purely transverse shock wave, leading only to shear without a change in volume, can exist in a nonlinear elastic medium only for a particular deformed state in front of the surface of discontinuity. This means that a shear shock wave will always simultaneously be a compressional wave. It was shown in [2] that in this case- the bulk deformations are of second order in comparison with shear deformations and for real materials they lead to an expansion of the medium. On the other hand, in [3] the self-modeling problem on the pure shear of an elastic half-space was considered, and it was shown that a centralizer shear wave also leads to expansion, for the same properties of the elastic medium. Therefore, one can obtain two solutions of the same self-modeling problem of the nonlinear dynamical theory of elasticity depending on the formulation of the problem. Self-modeling dynamical problems of the nonlinear theory of elasticity were considered in [1, 3-5] and shock waves in an elastic medium in [2, 6, 7].

In the present paper we formulate and present the numerical results of the self-modeling problem of the nonlinear dynamical theory of elasticity for the reflection of a plane longitudinal shock wave of constant intensity from a plane rigid boundary with an elastic medium. It is shown that for angles of incidence of the original shock wave which are less than a certain critical value (which depends on the wave intensity) two solutions of the problem are possible: a reflected quasitransverse shock wave or a centralized shear wave. For angles of incidence exceeding the critical value, the solution exists only for the reflected shock wave. The leading front of the bulk deformations is caused by the reflection of the shock wave from the rigid barrier and is a quasitransverse shock wave.

1. The system of equations describing the dynamical deformation of an elastic medium in a rectangular coordinate system in terms of the Euler variables has the form [8, 9]

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$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial W}{\partial e_{ik}}(\delta_{kj} - 2e_{kj}), \quad \sigma_{ij,j} = \rho\frac{dv_i}{dt},$$
$$v_i = \partial u_i/\partial t + v_j u_{i,j}, \; 2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j},$$
$$\rho/\rho_0 = \left(1 - 2I_1 + 2I_1^2 - 2I_2 - \frac{4}{3}I_1^3 + 4I_1I_2 - \frac{8}{3}I_3\right)^{1/2},$$
$$I_1 = e_{ii}, \; I_2 = e_{ij}e_{ji}, \; I_3 = e_{ij}e_{jk}e_{ki}. \tag{1.1}$$

In the expansion of the elastic potential $W(I_1, I_2, I_3)$ in a Taylor series with respect to the free state, we limit the expansion to terms of not higher than the third order in the deformation gradient tensor $u_{i,j}$. That is, we assume

$$W = \frac{1}{2}\lambda I_1 + 2\mu I_2 + lI_1I_2 + mI_1^3 + nI_3. \tag{1.2}$$

We shall consider the case of plane deformations. Then if we introduce the variables $u_1 = x_1u(\xi)$, $u_2 = x_1v(\xi)$, $\xi = (x_2 - st)/x_1$, it follows from (1.1) and (1.2) that

$$A_1u'' + A_2v'' = 0, \; A_3u'' + A_4v'' = 0, \tag{1.3}$$

where

$$A_j = a_0^j + a_1^j u + a_2^j u' + a_3^j v + a_4^j v' \quad (j = 1, 2, 3, 4),$$
$$a_0^1 = \xi^2 + \gamma_5 - T, \quad a_0^2 = a_0^3 = -\xi(\gamma_5 + \gamma_6), \quad a_0^4 = \xi^2\gamma_5 + 1 - T,$$
$$a_1^1 = \xi^2(\gamma_1 - 2) - 2\gamma_5 + \gamma_4 + 2T; \quad a_2^1 = \xi^3(2 - \gamma_1) + \xi(2\gamma_5 - 2\gamma_4 - \gamma_3),$$
$$a_3^1 = -\xi(1 + 2\gamma_4 + \gamma_5), \quad a_4^1 = \xi^2(\gamma_2 - 1 + 2\gamma_4 + \gamma_5) + \gamma_4 - \gamma_5 + T,$$
$$a_1^2 = \xi(2\gamma_6 + \gamma_5 - \gamma_2 - \gamma_4), \quad a_2^2 = \xi^2(\gamma_2 - 2\gamma_6 + 2\gamma_4 - \gamma_5 + 1) + \gamma_4 + \gamma_5 - T,$$
$$a_3^2 = \gamma_4(\xi^2 + 1), \quad a_4^2 = -\xi^3\gamma_4 + [2(\gamma_5 + \gamma_6 - \gamma_4) - \gamma_2 - 1].$$
$$a_1^3 = \xi(\gamma_6 - \gamma_4 - \gamma_2), \quad a_2^3 = \xi^2(\gamma_2 + 2\gamma_4 - \gamma_6),$$
$$a_3^3 = \xi^2(\gamma_4 + \gamma_5) + \gamma_4 + 1 - T, \quad a_4^3 = -\xi^3(\gamma_4 + \gamma_5) + \xi(\gamma_5 + 2\gamma_6 - 2\gamma_4 -$$
$$- \gamma_2 - 1 + T), \quad a_1^4 = \xi^2(\gamma_4 - \gamma_5) - 2 + \gamma_2 + 3T,$$
$$a_2^4 = -\xi^3(\gamma_4 - \gamma_5) + \xi(1 - 2\gamma_4 - \gamma_5 - \gamma_2 - T), \quad a_3^4 = -\xi(\gamma_3 + \gamma_4),$$
$$a_4^4 = \xi^2(2\gamma_4 - 2\gamma_5 + \gamma_3) + \gamma_1 - 2, \; T = s^2/G_0^2, \quad G_0 = [(\lambda + 2\mu)/\rho_0]^{\frac{1}{2}},$$
$$\gamma_1 = \frac{6(l + m + n)}{\lambda + 2\mu} - 7, \quad \gamma_2 = \frac{6m - 2l - 3\lambda}{\lambda + 2\mu}, \quad \gamma_5 = \frac{\mu}{\lambda + 2\mu},$$
$$\gamma_3 = \frac{l + \frac{3}{2}n}{\lambda + 2\mu} - 1, \quad \gamma_4 = \gamma_3 - 2\gamma_5, \quad \gamma_6 = \frac{\lambda}{\lambda + 2\mu}.$$

Let a plane shock wave Σ_i of constant intensity τ propagating in an undeformed elastic medium be incident at a certain angle α_1 on a plane rigid barrier L (Fig. 1). We show that this problem can be solved using the self-modeling variable ξ. We put $s = G_1 \sin\alpha_1$, where G_1 is the velocity of the shear wave. Then the position of the shear wave is determined by the value of the parameter $\xi = \xi_1 = \cot\alpha_i = \text{const}$, so that all of the boundary conditions of the problem are specified on the half-planes $\xi = \text{const}$. Therefore the problem can be solved in the framework of the self-modeling parameter.

It follows from (1.3) that the solution of the system of equations is trivial ($u' = \text{const}$, $v' = \text{const}$) if its determinant $A_1A_4 - A_2A_3 \neq 0$. The determinant can vanish for certain values of ξ which characterize the position of the reflected shock wave or in the interval of ξ corresponding to a reflected centralizer wave.

The presence of the rigid barrier causes a further compression of the medium. Perturbations leading to compression of the medium will propagate because of the reflected quasi-longitudinal shock wave Σ_2 ($\xi = \xi_2$) [2, 3, 7]; this picture is supported by the numerical calculations to be presented below. Additional shear deformations propagate in the medium only as a result of the reflected quasitransverse shock wave ($\xi_3 = \xi_4$) or the reflected centralized wave (the region between ξ_3 and ξ_4). It is shown below that depending on the initial parameters α_1 and τ, two cases can occur. In the regions corresponding to the trivial solution between ξ_1 and ξ_2 (region 1), between ξ_2 and ξ_3 (region 2), between ξ_4 and L (region 4) one can assume $u^{(i)} = a_i\xi + b_i$, $v^{(i)} = c_i\xi + d_i$ (i = 1, 2, 3, 4 gives the number of the

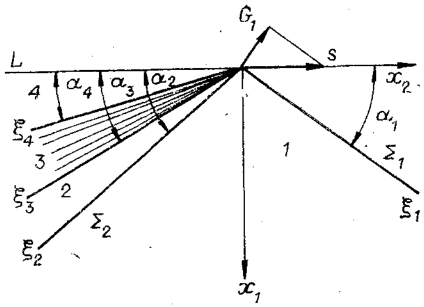

Fig. 1

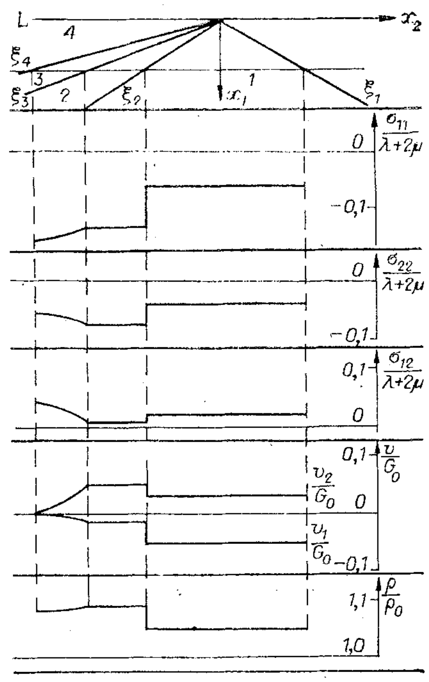

Fig. 2

region). Then

$$u_{1,1}^{(i)} = b_i,\ u_{1,2}^{(i)} = a_i,\ u_{2,1}^{(i)} = d_i,\ u_{2,2}^{(i)} = c_i,$$
$$v_1^{(i)} = -\frac{sa_i}{1 - b_i - c_i - a_i d_i + b_i c_i},\tag{1.4}$$
$$v_2^{(i)} = -s\frac{c_i - b_i c_i + a_i d_i}{1 - b_i - c_i - a_i d_i + b_i c_i},$$

where a_i, b_i, c_i, d_i are dimensionless constants. The stress and deformation will also be constants and can be calculated in terms of $u_{i,j}$ using (1.1) and (1.2).

2. The dynamical and kinematic conditions of compatibility on the shock wave discontinuity are written in the form [10]

$$[\sigma_{ij}]\nu_j = \rho^+(v_\nu^+ - G)[v_i],\quad [f] = f^+ - f^-,$$
$$[v_i] = u_{i,j}^+[v_j] + (v_\nu^- - G)\tau_i,\quad [u_{i,j}] = \tau_i\nu_j.\tag{2.1}$$

Here the plus and minus signs mean that the quantity in question is calculated in front of the shock wave and directly behind it, respectively, ν_j is the unit normal to the surface of discontinuity, and G is the velocity of the shock wave. Because we assume that Σ_1 has a constant intensity τ, applying (2.1) to Σ_1 gives

$$a_1 = d_1 = \tau\sin\alpha_1\cos\alpha_1,\ b_1 = -\tau\cos^2\alpha_1,\ c_1 = -\tau\sin^2\alpha_1,$$
$$G_1 = G_0(1 - \varkappa\tau)^{1/2},\quad \varkappa = -\frac{9}{2} + 3\frac{l + m + n}{\lambda + 2\mu},\quad \tau = \tau_i\nu_i.\tag{2.2}$$

The relations (2.2) determine all of the parameters of the deformed state in region 1 in terms of the known values of τ and α_1.

The continuity condition of the displacement on the shock wave Σ_2 has the form

$$(a_1 - a_2)\xi_2 + b_1 - b_2 = 0,\ (c_1 - c_2)\xi_2 + d_1 - d_2 = 0.\tag{2.3}$$

The conservation of momentum (2.1) on Σ_2 leads to the equations

$$-\left(\sigma_{11}^{(1)} - \sigma_{11}^{(2)}\right)\xi_2 + \sigma_{12}^{(1)} - \sigma_{12}^{(2)} = (1+\tau)\left\{\frac{G_1}{G_0}\frac{\tau}{1+\tau}(\sin\alpha_1 + \xi_2\cos\alpha_1)\right\}\left(\frac{v_1^{(1)}}{G_0} - \frac{v_1^{(2)}}{G_0}\right),$$
$$-\left(\sigma_{21}^{(1)} - \sigma_{21}^{(2)}\right)\xi_2 + \sigma_{22}^{(1)} - \sigma_{22}^{(2)} = (1+\tau)\left\{\frac{G_1}{G_0}\frac{\tau}{1+\tau}(\sin\alpha_1 + \xi_2\cos\alpha_1)\right\}\left(\frac{v_2^{(1)}}{G_0} - \frac{v_2^{(2)}}{G_0}\right).\tag{2.4}$$

In (2.4) $\sigma_{kj}^{(1)}$ are dimensionless stresses corresponding to $\lambda + 2\mu$. We assume that ξ_2 is known, then (2.3) and (2.4) form a system of four nonlinear algebraic equations for a_2, b_2,

c_2, d_2. We assume that $\sigma_{ij}^{(2)}$ and $v_i^{(2)}$ can be expressed in terms of a_2, b_2, c_2, d_2 according to (1.1), (1.2), and (1.4). Hence in region 2 the solution can be found as a function of the parameter ξ_2. In the case when the reflected shear excitations propagate in the medium because of a reflected centralized wave ($\xi_3 > \xi_4$) the formulation of the solution of (2.3) and (2.4) in terms of the equation

$$A_1A_4 - A_2A_3 = 0 \tag{2.5}$$

gives ξ_2 for a given ξ_3. The solution in the region between ξ_3 and ξ_4 is found by solving the Cauchy problem for the system of differential equations (2.5) and one of Eqs. (1.3). The boundary conditions for the Cauchy problem have the form

$$u'(\xi_3) = a_2,\ u(\xi_3) = a_2\xi_3 + b_2,\ v(\xi_3) = c_2\xi_3 + d_2.$$

We consider the solution of the Cauchy problem when u' and v' are zero. By varying ξ_2 one can get u' and v' equal to zero simultaneously. This case corresponds to the boundary conditions $u = v = 0$ on L.

If the leading front of the reflected shear excitations is a quasitransverse shock wave ($\xi_3 = \xi_4$), then the continuity condition of the displacement and the conservation of momentum on the shock are written from (2.1) in the form

$$\begin{gathered} a_2\xi_3 + b_2 - b_4 = 0,\ c_2\xi_3 + d_2 - d_4 = 0, \\ -\left(\sigma_{11}^{(2)} - \sigma_{11}^{(4)}\right)\xi_3 + \sigma_{12}^{(2)} - \sigma_{12}^{(4)} = \left(1 - b_2 - c_2\right)\left(-\xi_3 v_1^{(2)} + v_2^{(2)} - s\right) v_1^{(2)}/G_0^2, \\ -\left(\sigma_{21}^{(2)} - \sigma_{21}^{(4)}\right)\xi_3 + \sigma_{22}^{(2)} - \sigma_{22}^{(4)} = \left(1 - b_2 - c_2\right)\left(-\xi_3 v_1^{(2)} + v_2^{(2)} - s\right) v_2^{(2)}/G_0^2. \end{gathered} \tag{2.6}$$

In (2.6) the boundary conditions $a_4 = c_4 = v_1^{(4)} = v_2^{(4)} = 0$ are taken into account, and the density in region 2 is calculated according to the equation of continuity from (1.1). If in (2.4) and (2.6) we take into account (1.4) and (2.2), and also that $G_2 = G_1\{(1 + \cot^2\alpha_1)/(1 + \xi_2^2)\}^{1/2}$, then we obtain a system of eight nonlinear algebraic equations for ξ_2, $\xi_3 = \xi_4$, a_2, b_2, c_2, d_2, b_4, d_4.

We note that it is not possible to determine in advance if the reflected shear excitations are propagated in the medium because of a shock or centralized wave.

3. Numerical calculations were carried out on the basis of the discussion given above. The parameters in the problem were varied in the range $5° \leqslant \alpha_1 \leqslant 60°$, which corresponds to $11.43 \geqslant \xi \geqslant 0.577$ and $0.1 \geqslant \tau \geqslant 0.01$. The elastic constants were chosen in the form [11] $\lambda/(\lambda + 2\mu) = 0.374$, $\mu/(\lambda + 2\mu) = 0.313$, $l/(\lambda + 2\mu) = -1.24$, $m/(\lambda + 2\mu) = -0.412$, $n/(\lambda + 2\mu) = -0.663$.

The basic qualitative results of the calculations are: If α_1 varies between 5° and α_*, where α_* depends on τ and the constants of the material, then solutions with a reflected quasitransverse shock wave and a reflected centralized wave are both possible. When $\alpha_1 > \alpha_*$, the solution is only found with a reflected quasitransverse shock wave. The limiting value $\alpha_1 = \alpha_*$ decreases with increasing τ; when $\tau = 0{,}01$ $\xi_* = 0{,}936$ ($\alpha_* = 46{,}9°$), when $\tau = 0{,}03$ $\xi_* = 1{,}014$ ($\alpha_* = 44.6°$), for $\tau = 0{,}05$ $\xi_* = 1{,}208$ ($\alpha_* = 39{,}6°$). Hence the solution of the problem is not unique when $\alpha_1 < \alpha_*$. We discuss a typical result of the calculations for $\alpha_1 = 30°$, $\tau = 0.05$. When we have reflection of a quasitransverse shock wave, $\alpha_2 = 31.85°$, $\sigma_{11}^{(2)} = -0{,}1344$, $\sigma_{12}^{(2)} = 0{,}0072$, $\sigma_{22}^{(2)} = -0{,}0745$, $v_1^{(2)}/G_0 = -0{,}0069$, $v_2^{(2)}/G_0 = 0{,}0509$, $\rho^{(2)}/\rho_0 = 1{,}0883$, $\alpha_3 = \alpha_4 = 16{,}30°$ ($\xi_3 = \xi_4$), $\sigma_{11}^{(4)} = -0.153$, $\sigma_{12}^{(4)} = 0{,}0467$, $\sigma_{22}^{(4)} = -0.0576$, $\rho^{(4)}/\rho_0 = 1{,}0777$ If the reflected shear excitations propagate because of a centralized wave, then these parameters will have the following values: $\alpha_2 = 31{,}5°$, $\sigma_{11}^{(2)} = -0{,}1266$, $\sigma_{12}^{(2)} = 0{,}0092$, $\sigma_{22}^{(2)} = -0{,}0701$, $v_1^{(2)}/G_0 = -0{,}0114$, $v_2^{(2)}/G_0 = 0{,}0483$, $\rho^{(2)}/\rho_0 = 1{,}0845$, $\alpha_3 = 17{,}34°$, $\alpha_4 = 15{,}72°$, $\sigma_{11}^{(4)} = -0{,}1524$, $\sigma_{12}^{(4)} = 0.0450$ $\sigma_{22}^{(4)} = -0{,}0574$, $\rho^{(4)}/\rho_0 = 1{,}0784$.

For $\alpha < \alpha_*$, when the intensity τ of the incident shock wave increases, α_2 increases and we always have $\alpha_2 > \alpha_1$. The stress in the region 2 increases sharply with increasing τ; for example, when $\alpha_1 = 30°$, $\tau = 0{,}01$ $\sigma_{11}^{(2)} = -0{,}0167$, $\sigma_{12}^{(2)} = 0{,}0009$, $\sigma_{22}^{(2)} = -0{,}0102$, $\rho^{(2)}/\rho_0 = 1.0172$, for $\tau = 0.05$ the values of the dimensionless stresses (corresponding to $\lambda + 2\mu$) and the relative densities are as given above. The numerical results show that in all cases reflection of a quasilongitudinal shock wave ($\xi = \xi_2$) is a wave of compression $\rho^{(2)}/\rho_0 > 1$. The centralized wave leads to an expansion of the medium $\rho^{(4)}/\rho^{(2)} < 1$. In region 3 of the centralized wave, the functions σ_{ij}, v_i, ρ/ρ_0 are monotonic (Fig. 2 where $\tau = 0.05$, $\xi_1 = 1.732$, $\xi_2 = -1.632$, $\xi_3 = -3.248$, $\xi_4 = -3.618$). In this case the relative density decreases, v_1/G_0 increases and remains negative, v_2/G_0 decreases, the stresses σ_{11} and σ_{12} increase in absolute value, and σ_{22} decreases. The width of the centralized wave $h = |\xi_3 - \xi_4|$ increases with increasing intensity τ of the incident shock wave; for example, when $\alpha_1 = 30°$ and $\tau = 0.01$, $h = 0.05$; when $\tau = 0.05$, $h = 0.33$. As the angle of incidence α_1 of the shock wave increases, the width of the centralized wave increases from $h = 0.004$ ($\alpha_1 = 15°$) to $h = 0.06$ ($\alpha_1 = 45°$) for $\tau = 0.01$.

If $\alpha_1 > \alpha_*$, the solution of the problem (as noted above) simplifies and is found by solving eight nonlinear algebraic equations. The calculation shows that Σ_2 is always a quasi-longitudinal compressional shock wave, the quasitransverse shock wave ($\xi = \xi_3 = \xi_4$) in all cases leads to an expansion of the medium; this has also been noted in the theoretical treatments [2]. We show typical numerical results for the change in the position of the reflected shock wave as a function of the change in the intensity of the incident wave ($\alpha_1 = 60°$):

$$\tau = 0.01\ \alpha_2 = 60.10°,\ \alpha_3 = -\operatorname{arctg} \xi^{-1} = 29.02°;$$
$$\tau = 0.03\ \alpha_2 = 62.27°,\ \alpha_3 = 28.84°;$$
$$\tau = 0.05\ \alpha_2 = 72.00°,\ \alpha_3 = 28.83°.$$

We did not consider the case $\alpha_1 > 60°$ in the calculations since irregular reflection is possible for large values of α_1.

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